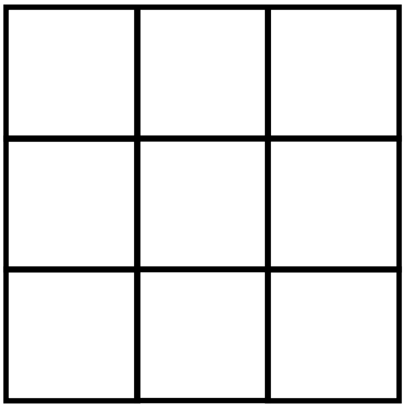
MATHS CRAFT new zealand

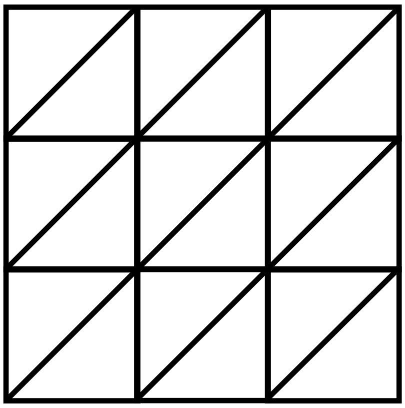
Penrose Tilings

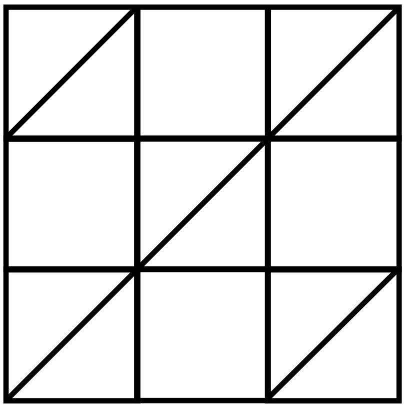
Tiling the plane means using shapes to cover a flat surface without any gaps. Often these tilings repeat a pattern over and over. For a long time, mathematicians wondered whether a pattern without repeats was possible. A **Penrose tiling** is an example of a tiling which does not repeat, no matter how big the surface. It is composed of just two simple shapes called tiles, and this is surprising because no-one suspected that a non-repeating tiling could be done with just two tiles. Penrose tilings have many other surprising and beautiful properties.

Tiling the Plane

In a general tiling, the tiles can all be identical, or a combination of different shapes. For example, the plane can be tiled with identical squares, as on a chessboard (below left), with triangles (by cutting each chessboard square diagonally in half, below centre), or with a combination of the two (below right).



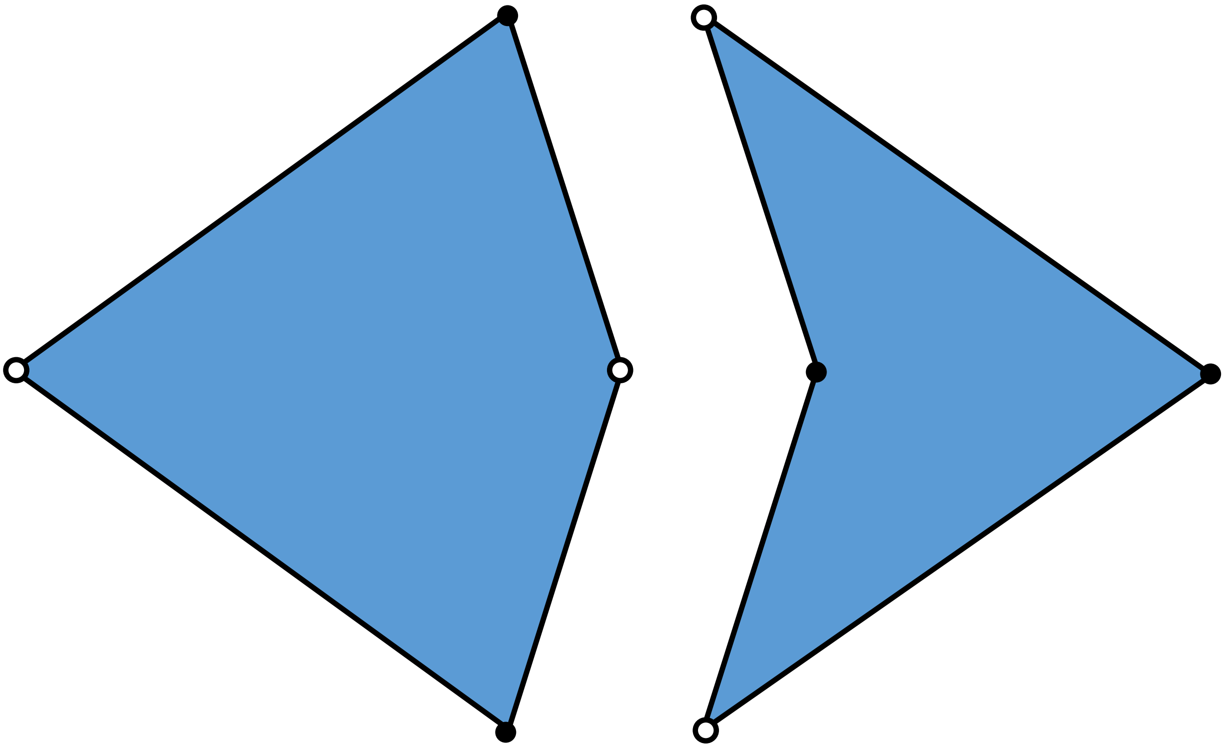




The chessboard is composed of squares arranged in a regularly repeating pattern. (A pattern is an arrangement of tiles.) Tilings like this whose patterns repeat periodically are called *periodic*. Not all tilings are periodic. For example, a spiral pattern of identical triangles tiles the plane, but there is just a single spiral pattern which is not repeated. Tilings which are not periodic are called *nonperiodic*.

Triangles can be arranged both periodically and nonperiodically. The same is true of squares, or a combination of squares and triangles. Until the 1960s, all known nonperiodic tilings were made up of tiles which could also be arranged into periodic tilings. Then a set of tiles which can only be arranged into nonperiodic tilings was discovered; mathematicians call this an *aperiodic* set of tiles. This set contains 20,000 tiles! But the number was rapidly reduced until in 1974, Roger Penrose found an aperiodic set of just two tiles. These are now known as the “kite” and “dart”, and are shown below.

Rodger Penrose is a physicist with a love for recreational mathematics, which is shared by his geneticist father. Together, Penrose junior and senior invented the impossible drawing of the “Penrose staircase”, in which figures forever ascend or descend an impossibly looped staircase, and the “impossible triangle” which you can learn to draw with our *Draw an Impossible Triangle* handout.



Kite and Dart

The kite (above left) is a four-sided shape whose internal angles are 72, 72, 72, and 144 degrees, while the dart (above right) is a four-sided shape whose angles are 36, 72, 36, and 216 degrees. There is a strict rule on how these shapes are allowed to be put together in a tiling, and following this rule guarantees a nonperiodic tiling of the plane. The rule can be represented in different ways, but one way is to colour the corners of each tile either black or white as shown above. The arrangement rule states: **place the corners of shapes together so that like-coloured corners go together**.

A good way to build up a nonperiodic tiling following this rule is to start with a single tile, and arrange other tiles around it, expanding radially. With each tile you add to the pattern you can choose a dart or a kite, and while the choice is sometimes forced, at other times it is not. A good approach is to go around the boundary laying out all the forced tiles first. Then experiment with the unforced tiles, which might need some correction if you later encounter a place where no tile can be added while following the rule. Continue in this way, and you will tile the plane nonperiodically.

Further Explorations

There is much more you can discover about Penrose tilings. A good place to start is the article which introduced Penrose tilings to the world, written by the great populariser of mathematics, Martin Gardner. This article is freely available at http://bit.ly/2vsUZoc. For example, any part of a Penrose tiling will repeat infinitely often, just not in a periodic way! And can you beat Penrose’s record of a set of two aperiodic tiles – can you find a single tile which tiles the plane nonperiodically? No-one has been able to do this, or even knows if it is possible!